Old UNM Algebra Qualifier Problems

Based on Chapters 1 and 2 of Dummit–Foote

Fifteen-year collection: January 2012–August 2023

Scope. This packet uses the third edition of Dummit and Foote, *Abstract Algebra*. Chapter 1 is "Introduction to Groups" and Chapter 2 is "Subgroups." The first section contains problems that can reasonably be attempted after those two chapters. The second section records closely related problems whose standard solutions use some later material, such as cosets and Lagrange's theorem, quotient groups, conjugacy classes, or more developed group actions.

I. Core problems from Chapters 1–2

1. January 2012 Qualifier, Problem 3

Prove that the group with generators a, b and relations

$$a^3 = b^2 = e, \qquad ab = ba^2,$$

is isomorphic to the symmetric group S_3.

2. August 2012 Qualifier, Problem 2

Prove that the group defined by generators a, b, c, d and relations

$$adb = b^2a, \qquad a^3 = c, \qquad b^2 = c, \qquad d^2 = c$$

is infinite and noncommutative.

3. January 2014 Qualifier, Problem 1

Let G be a group. Prove that G is abelian if and only if the map

$$f : G \longrightarrow G, \qquad f(g) = g^2,$$

is a homomorphism.

Note: Solving this also solves **January 2019 Qualifier, Problem 1**.

4. January 2014 Qualifier, Problem 2

Show that the diagonal subgroup

$$D = \{(a, a) : a \in S_3\} \subseteq S_3 \times S_3$$

is not normal.

5. August 2016 Qualifier, Problem 2

Let G be a group. Show that the set of inner automorphisms of G is a normal subgroup of the group of automorphisms of G.

Note: For $g \in G$, define $c_g : G \to G$ by $c_g(x) = gxg^{-1}$. The set $\mathrm{Inn}(G) = \{c_g : g \in G\}$ is called the set of **inner automorphisms** of G.

6. January 2018 Qualifier, Problem 7

Suppose that, for some group G, $|\operatorname{Aut}(G)| = 1$. Show that G is abelian and every element of G has order at most 2.

Note: This problem is repeated in **August 2022 Qualifier, Problem 1** and in **August 2023 Qualifier, Problem 2**.

7. August 2020 Qualifier, Problem 10

Prove that the group with generators a, b and relations

$$a^2 = b^2 = e$$

is infinite.

8. January 2021 Qualifier, Problem 3

Prove that for every noncyclic subgroup G of $(\mathbb{R}, +)$ and every pair of real numbers $a < b$, there exists $g \in G$ such that $a < g < b$.

9. January 2023 Qualifier, Problem 2

Show that S_A acts on $\mathcal{P}(A)$, the power set of A, via

$$\sigma \cdot B = \{\sigma(b) : b \in B\}.$$

When $A = \{1, 2, 3, 4\}$, determine the following:

(a) the orbit of $\{1, 2\}$;

(b) the stabilizer of $\{1, 3, 4\}$;

(c) the fixed set of $(12)(34)$.

II. Closely related or borderline problems

1. August 2012 Qualifier, Problem 3

Let σ and τ be distinct 3-cycles in S_5. Show that $\langle \sigma, \tau \rangle$ is either isomorphic to A_4 or is A_5.

Correction to the printed problem. The assertion is false as printed. If $\sigma = (123)$ and $\tau = (132) = \sigma^{-1}$, then the cycles are distinct but $\langle \sigma, \tau \rangle = \langle \sigma \rangle \cong C_3$. The correct classification is

$$\langle \sigma, \tau \rangle \cong C_3, \quad A_4, \quad \text{or} \quad A_5.$$

2. August 2014 Qualifier, Problem 1

A finite group G acts on itself by conjugation. Determine all possible G if this action yields precisely three distinct orbits.

3. August 2014 Qualifier, Problem 2

Let G be the group of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in (\mathbb{Z}/p\mathbb{Z})^\times,\ b \in \mathbb{Z}/p\mathbb{Z} \right\}.$$

Find all normal subgroups of G.

4. August 2016 Qualifier, Problem 1

Prove that if G and H are two finite groups with coprime orders, then there is no nontrivial homomorphism between them.

5. January 2017 Qualifier, Problem 3

Let $K \le H \le G$ be a chain of subgroups.

(a) If H is characteristic in G and K is characteristic in H, show that K is characteristic in G.

(b) Show that if "characteristic" is replaced by "normal" throughout the preceding statement, then the amended statement is false.

Note: A subgroup $H \le G$ is **characteristic** if $\alpha(H) = H$ for every $\alpha \in \mathrm{Aut}(G)$.

6. January 2019 Qualifier, Problem 3

Suppose G acts on both X and Y. A map $\varphi : X \to Y$ is called a G-isomorphism if it is a bijection and

$$g\varphi(x) = \varphi(gx)$$

for every $g \in G$ and $x \in X$. Let G act transitively on X, fix $x_0 \in X$, let H be the stabilizer of x_0, and let L be the set of left cosets of H. Show that X and L are G-isomorphic.

7. August 2019 Qualifier, Problem 4

Suppose M is a maximal proper subgroup of a group G. Prove that either $N_G(M) = M$ or $N_G(M) = G$. Deduce that if M is a maximal subgroup of a finite group G which is not normal, then the number of nonidentity elements contained in the conjugates of M is

$$(|M| - 1)(G : M).$$

Correction to the printed problem. The second part should be as follows: Assume M is a maximal subgroup of a finite group G which is not normal, and distinct conjugates of M intersect only in the identity. Then

$$\left| \left(\bigcup_{g \in G} gMg^{-1} \right) \setminus \{1\} \right| = (|M| - 1)(G : M).$$

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8. August 2023 Qualifier, Problem 1

Suppose H is a subgroup of a finite group G and

$$\gcd(|H|, (G : H)) = 1.$$

Is H necessarily the unique subgroup of G of order $|H|$? If the answer is yes, give a proof; otherwise, give a counterexample.

Source

University of New Mexico, Department of Mathematics and Statistics, Past Qualifying Exams – Algebra. Individual exam dates and problem numbers are recorded above.